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Some examples of exponentially harmonic maps

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Abstract

The aim of this paper is to study some examples of exponentially harmonic maps. We study such maps first on flat Euclidean and Minkowski spaces and then on Friedmann–Lemaître universes. We also consider some new models of exponentially harmonic maps which are coupled with gravity which happen to be based on a generalization of the Lagrangian for bosonic strings coupled with dilatonic field.

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1. Introduction

Exponentially harmonic maps were introduced by James Eells and studied by Eells and Lemaire [1]. These maps generalize the usual harmonic maps [2] in the following sense. Let (M, g) and (N, h) be two Riemannian manifolds and $\phi : M \to N : x \to \phi(x)$, a smooth map. An exponentially harmonic map is then an extremal of the following functional:

$$E(\phi) = \int_M \exp(e(\phi))\, \mathrm{d}\mu(\phi) \tag{1}$$

where $\mathrm{d}\mu(\phi)$ is the Riemannian volume element and

$$e(\phi) = \frac{1}{2}\frac{\partial\phi^a}{\partial x^\mu}\frac{\partial\phi^b}{\partial x^\nu} g^{\mu\nu} h_{ab} \tag{2}$$

is the so-called 'energy density' of the map ϕ. In local coordinates (x^μ) and (ϕ^a), the equation for exponentially harmonic maps, which derives from the variation of the functional (1), is

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$$\exp(e(\phi)) \left\{ g^{\alpha\beta} \left(\frac{\partial^2 \phi^a}{\partial x^\alpha \partial x^\beta} - \Gamma^{\gamma(M)}_{\alpha\beta} \frac{\partial \phi^a}{\partial x^\gamma} + \Gamma^{a(N)}_{bc} \frac{\partial \phi^b}{\partial x^\alpha} \frac{\partial \phi^c}{\partial x^\beta} \right) \right.$$
$$+ g^{\alpha\mu} g^{\beta\nu} h_{bc} \frac{\partial \phi^a}{\partial x^\mu} \frac{\partial \phi^c}{\partial x^\nu} \frac{\partial^2 \phi^b}{\partial x^\alpha \partial x^\beta} - g^{\alpha\mu} g^{\beta\nu} h_{bc} \Gamma^{\gamma(M)}_{\alpha\beta} \frac{\partial \phi^a}{\partial x^\mu} \frac{\partial \phi^b}{\partial x^\nu} \frac{\partial \phi^c}{\partial x^\gamma}$$
$$\left. + g^{\alpha\beta} g^{\mu\nu} h_{bc} \Gamma^{b(N)}_{de} \frac{\partial \phi^d}{\partial x^\alpha} \frac{\partial \phi^e}{\partial x^\mu} \frac{\partial \phi^c}{\partial x^\nu} \frac{\partial \phi^a}{\partial x^\beta} \right\} = 0 \tag{3}$$

where $\Gamma^{\gamma(M)}_{\alpha\beta}$ and $\Gamma^{a(N)}_{bc}$ are the Christoffel symbols of the Levi-Civita connection on M and N. This equation involves, as a particular case, the equation of the usual harmonic maps. If we drop the $\exp(e(\phi))$ factor and restrict ourselves to the first three terms we get

$$g^{\alpha\beta} \left(\frac{\partial^2 \phi^a}{\partial x^\alpha \partial x^\beta} - \Gamma^{\gamma(M)}_{\alpha\beta} \frac{\partial \phi^a}{\partial x^\gamma} + \Gamma^{a(N)}_{bc} \frac{\partial \phi^b}{\partial x^\alpha} \frac{\partial \phi^c}{\partial x^\beta} \right) = 0 \tag{4}$$

which is the field equation for a usual harmonic map (which is nothing but a non-linear sigma model in the physicist's language). When $N = \mathbb{R}$, (4) is simply written as

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial \phi}{\partial x^\beta} \right) = 0 \tag{5}$$

which is the Laplace equation in local coordinates. Following the work of Lemaire, it is important to note, however, that the properties of the exponentially harmonic maps are very different from those of the usual harmonic maps. This comes from the fact that the functional (1) changes completely when we perform a conformal shift on the metric h: $h \to \lambda h$. Then (1) becomes

$$E_\lambda(\phi) = \int_M [\exp(e(\phi))]^\lambda \, \mathrm{d}\mu(\phi) \qquad \lambda = \text{constant}. \tag{6}$$

In the case of the usual harmonic maps this metric shift has no influence on the harmonic map equations, the functional is simply multiplied by a constant.

In the sequel we will study some particular cases of (3) even when M is noncompact and (1) unbounded. In this case, equation (3) is taken as the definition of what we call an exponentially harmonic map. Following Eells and Lemaire, we will consider the energy–momentum tensor associated with ϕ:

$$T_{\mu\nu}(\phi) = \exp(e(\phi)) \left(g_{\mu\nu} - \frac{\partial \phi^a}{\partial x^\mu} \frac{\partial \phi^b}{\partial x^\nu} h_{ab} \right) \tag{7}$$

which is conserved ($g^{\alpha\beta} \nabla_\alpha T_{\alpha\beta}(\phi) = 0$) when ϕ is a solution of (3).

It is worth noting that some particular cases of (3) were studied by mathematicians in the context of the theory of elliptic partial differential equations. Guilbarg and Trudinger [3] quote, for example, the equation

$$\Delta\phi + \beta \sum_{i,j=1}^{n} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^j} \frac{\partial^2 \phi}{\partial x^i \partial x^j} = 0 \qquad \phi : U \subset \mathbb{R}^n \to \mathbb{R} \tag{8}$$

deriving from the variational principle: $\delta \int_{\mathrm{U}} \exp\left(\frac{\beta}{2} \sum_{i=1}^{n} \left(\frac{\partial \phi}{\partial x^i}\right)^2\right) \mathrm{d}x^1 \cdots \mathrm{d}x^n$.

Let us note finally that, if we want to make some contact with physics, we have to modify (1) as follows:

$$E'_\lambda = \int_M (\exp(\lambda e(\phi)) - 1) \, \mathrm{d}\mu(\phi)$$
$$\approx \lambda \int_M \left[\left(\frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi_a \right) + \frac{\lambda}{2} \left(\frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi_a \right)^2 + \cdots \right] \mathrm{d}\mu(\phi) \tag{9}$$

which, if we interpret (ϕ^a) as a spin vector, looks like the Hamiltonian for the continuum limit of a Heisenberg model. When λ is small enough, the variational principle $\delta E'_\lambda = 0$ leads to equations which approximate those of the usual harmonic maps (sigma models). The equations derived from E'_λ can be obtained directly from (3) via the shift $\phi \to \sqrt{\lambda}\phi$ $(\lambda > 0)$.

2. Exponential harmonic maps on flat spaces

In order to get some feeling about the solutions of (3), we study here scalar exponentially harmonic maps on two-dimensional Euclidean and Minkowskian manifolds:

$$\phi : E \to \mathbb{R} \qquad E = \mathbb{R}^2 \quad \text{or} \quad \mathbb{R}^{1,1}.$$

2.1. $E = \mathbb{R}^2$

Let $(x^\mu) = (x, y)$, $\phi_{x^\mu} = \partial\phi/\partial x^\mu$ and $\phi_{x^\mu x^\nu} = \partial^2\phi/\partial x^\mu \partial x^\nu$. Then, equation (3) becomes

$$\left(1 + \phi_x^2\right)\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + \left(1 + \phi_y^2\right)\phi_{yy} = 0. \tag{10}$$

We are searching for solutions of the following form: $\phi(x, y) = F(x) + G(y)$. We are thus led to the equations

$$\left[1 + (F_x)^2\right] F_{xx} = -\left[1 + (G_y)^2\right] G_{yy} = l \tag{11}$$

where l happens to be an arbitrary positive real constant. $l = 0$ leads, of course, to trivial solutions $\phi(x, y) = ax + by$. If $l > 0$, let $p = F_x$. This leads to the system

$$F_x = p(x) \tag{12}$$

$$p_x = \frac{l}{1 + p^2}. \tag{13}$$

Then (13) gives

$$x = \frac{1}{l}\left(\frac{1}{3}p^3 + p - c\right) \qquad c = \text{constant}. \tag{14}$$

Let us derive (14) with respect to F. This gives (with $k_1 = \text{constant}$).

$$\frac{\mathrm{d}x}{\mathrm{d}F} = \frac{1}{F_x} = \frac{1}{l}(1 + p^2)\frac{\mathrm{d}p}{\mathrm{d}F} \qquad \text{and} \qquad p^4 + 2p^2 - 4l(F + k_1) = 0. \tag{15}$$

The elimination of p from (14) and (15) gives

$$F(x) = \frac{1}{4l}\{[S_+(x; l; c) + S_-(x; l; c)]^4 + 2[S_+(x; l; c) + S_-(x; l; c)]^2\} - k_1 \tag{16}$$

$$p = S_+(x; l; c) + S_-(x; l; c) \tag{17}$$

where $S_\pm(x; l; c) = \left\{\frac{3}{2}(c + lx) \pm \left(1 + \frac{9}{4}(c + lx)^2\right)^{1/2}\right\}^{1/3}$. Similarly we get the solution $G(y)$. A solution of (10) can thus be written with the above notation as

$$G(y) = -\frac{1}{4l}\{[S_+(y; -l; c') + S_-(y; -l; c')]^4 + 2[S_+(y; -l; c') + S_-(y; -l; c')]^2\} - k_2$$

which leads to the solution we look for.

If we write $q(y) = S_+(y; -l; c') + S_-(y; -l; c')$, then we can represent $\phi(x, y)$ in the following parametric form:

$$x = \frac{1}{l}\left(\frac{p^3}{3} + p - c\right) \qquad y = \frac{-1}{l}\left(\frac{q^3}{3} + q - c'\right)$$

$$\phi(x, y) = \frac{1}{4l}(p^4 + 2p^2 - q^4 - 2q^2) + \text{constant.}$$

It is also possible to solve (10) by the so-called hodograph method [4]. Let $u = \phi_x$, $v = \phi_y$ and $x = x(u, v)$, $y = y(u, v)$. If $J = x_u y_v - x_v y_u \neq 0$, we get immediately

$$u_x = y_v/J \qquad v_x = -y_u/J$$

$$u_y = -x_v/J \qquad v_y = x_u/J.$$

We know that if $v_x = u_y$ then $x_v = y_u$. Thus, there exists a function $f(u, v)$ such that $x = f_u$ and $y = f_v$. Equation (10) can be written as

$$(1 + u^2)f_{vv} - 2uvf_{(uv)} + (1 + v^2)f_{uv} = 0 \tag{18}$$

or, using polar coordinates, $u = r\cos\theta$, $v = r\sin\theta$:

$$f_{rr} + \left(r + \frac{1}{r}\right) f_r + \left(1 + \frac{1}{r^2}\right) f_{\theta\theta} = 0 \tag{19}$$

which can be solved by factorizing: $f(r, \theta) = R(r)T(\theta)$. We now have to solve the two equations (with an arbitrary constant a)

$$R_{rr} + \left(r + \frac{1}{r}\right) R_r - a^2\left(1 + \frac{1}{r^2}\right) R = 0 \tag{20}$$

$$T_{\theta\theta} + a^2 T = 0. \tag{21}$$

If we are interested in periodic solutions, we set $T(\theta) = A\cos(a\theta) + B\sin(a\theta)$. Thus (20) can be reduced to the normal form by the following functional change:

$$R \mapsto \widehat{R} \qquad R(r) = \frac{1}{\sqrt{r}}\exp\left(-\frac{r^2}{4}\right)\widehat{R}(r).$$

Equation (20) then becomes

$$\widehat{R}_{rr} + \left(-(1 + a^2) + \left(\frac{1 - 4a^2}{4}\right)\frac{1}{r^2} - \frac{r^2}{4}\right)\widehat{R}. \tag{22}$$

This equation can be reduced to the Whittaker equation [5]. Writting $\widehat{R}(r) = r^{-1/2}M(r^2/2)$, then the function $M(\varepsilon)$ satisfies the Whittaker equation

$$M(\varepsilon) = M_{\alpha,\mu/2}(\varepsilon) \qquad \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}M_{\alpha,\mu/2} + \left(\frac{-1}{4} + \frac{\alpha}{\varepsilon} + \frac{1 - \mu^2}{4\varepsilon^2}\right)M_{\alpha,\mu/2} = 0$$

with $\alpha = \pm\frac{1}{2}(a^2 + 1)$, $\mu = \pm a$. This solution can be expressed in terms of the confluent hypergeometric function ${}_1F_1$ which leads to the solution of (20) written as

$$R(r) = \frac{1}{2^{(a+1)/2}}\,\mathrm{e}^{-r^2}r^a\,{}_1F_1\left(1 + \frac{a}{2} + \frac{a^2}{2}, 1 + a, r^2/2\right).$$

If $a > 0$, this solution is regular at the origin. We know that ${}_1F_1(m, m, x) = \exp(x)$. Then, for $a = 1$, we get $R(r) = r/2$, which is a solution of (20). Nevertheless, this trivial solution (and related to the case $a = 0$) leads to some problems because, if we set $T(\theta) = \cos\theta$, $f(r, \theta) = r\cos\theta$, then $f_v = 0$, $J = 0$.

2.2. $E = \mathbb{R}^{1,1}$

The computations are similar to the preceding case. Equation (3) leads to

$$\left(1+\phi_x^2\right)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} - \left(1-\phi_y^2\right)\phi_{yy} = 0.$$

The solutions of the form $\phi(x, y) = F(x) + F(y)$ can be written with a parametric representation:

$$x = \frac{1}{\lambda}\left(\frac{p^3}{3} + p - c\right) \qquad y = \frac{1}{\lambda}\left(\frac{-q^3}{3} + q - c'\right)$$

$$\phi(x, y) = \frac{1}{4\lambda}(p^4 + 2p^2 - q^4 + 2q^2) + \text{constant}.$$

3. Exponentially harmonic maps on Friedmann–Lemaître universe

Let M be a Friedmann–Lemaître (FL) universe endowed with the following metric:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right). \tag{23}$$

We consider exponentially harmonic maps: $\phi : M \to \mathbb{R}$ which are, for physical reasons explained above, extremals of the functional E'_λ. They satisfy a modified version of equation (3) which is written in this case:

$$\ddot{\phi}(1 + \lambda\dot{\phi}^2) + 3\frac{\dot{R}}{R}\dot{\phi} = 0 \qquad (\dot{\phi} = \phi_t, \dot{R} = R_t) \tag{24}$$

if we restrict ourselves to $\phi = \phi(t)$. This gives

$$R^3(t) = \frac{a}{|\dot{\phi}|}\exp\left(-\frac{\lambda}{2}\dot{\phi}^2\right) \qquad (a \text{ is a positive constant}). \tag{25}$$

Let us take, for example, $R(t) = R_o\left(\frac{t}{t_o}\right)^{2/3}$, i.e., M is a Euclidean FL-universe (without cosmological constant) and $t_o = 2/3h_o$ where H_o is the present value of the Hubble constant $(\dot{R}/R|_{t=t_o} = H_o)$. We get the field equation

$$|\dot{\phi}|\exp\left(\frac{\lambda}{2}\dot{\phi}^2\right) = 1/bt^2 \qquad \left(b = R_o^3\big/at_o^2\right).$$

(i) When $t \to \infty$ this gives $\phi(t) \to$ constant.
(ii) When $t \to 0$ and λ is small, $\phi(t) \approx \phi_o \pm 1/bt$ which is not regular at $t = 0$ (see figure 1).

When $\lambda \approx 0$ we recover the well-known theory of harmonic maps coupled with gravity studied by Hirschman and Schimming [6], Hughes *et al* [7], Lemaire and Vanderwinden [8].

4. Exponentially harmonic maps on an FL-universe coupled with gravity

Let us start now with the following action, describing a gravitational field coupled with an exponentially scalar field:

$$S(\phi) = -\frac{1}{2\kappa}\int\sqrt{-g}\,\mathrm{d}^4x\left\{\left(R - \exp\left(\frac{\lambda}{2}\partial_\alpha\phi\partial^\alpha\phi\right) - \Lambda\right) + \mathcal{L}_{\text{mat}}\right\} \tag{26}$$

κ is a coupling constant, Λ is a modified cosmological constant: $\Lambda = 2\kappa(2\Lambda_0 - 1)$ with Λ_0 is the usual cosmological constant, $\mathcal{L}_{\text{mat}}$ is the Lagrangian density for matter and $\phi : M \to \mathbb{R}$ is a

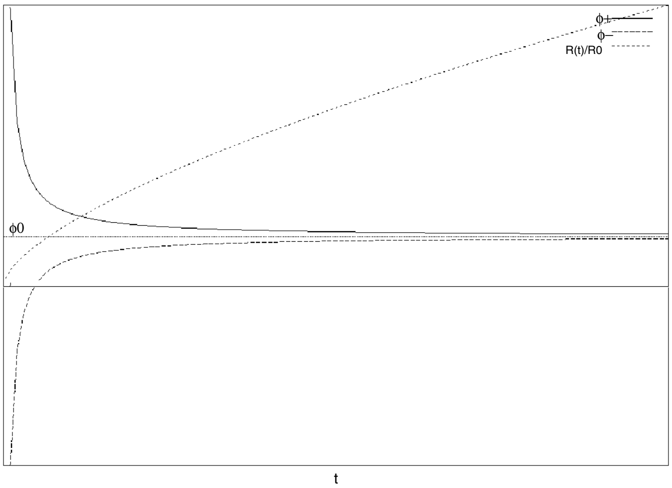

Figure 1. Behaviour of the expansion factor $R(t)$ and the exponentially harmonic map $\phi(t)$ on an Euclidean Friedmann–Lemaître spacetime ($\lambda \approx 0$; $\Lambda = 0$ and $\phi_0 > 0$).

scalar field defined on a four-dimensional spacetime. The variation of $S(\phi)$ leads to Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{2}\left\{g_{\mu\nu}\left(-\mathrm{e}^{\frac{\lambda}{2}\partial_\alpha\phi\partial^\alpha\phi} - \Lambda\right) + \lambda\partial_\mu\phi\partial_\nu\phi\,\mathrm{e}^{\frac{\lambda}{2}\partial_\alpha\phi\partial^\alpha\phi}\right\} + \kappa T^{(\mathrm{mat})}_{\mu\nu}$$

where $T^{(\mathrm{mat})}_{\mu\nu}$ is the usual energy–momentum tensor for matter. The variation with respect to ϕ gives a field equation which is very similar to (3). Let us assume that $\phi = \phi(t)$ and M is a Friedmann–Lemaître universe. Then the field equations can be written as

$$3\left(\frac{\dot{R}}{R}\right)^2 + 3\frac{k}{R^2} = \rho - \frac{1}{2}\,\mathrm{e}^{\frac{\lambda}{2}\dot{\phi}^2}(1 - \lambda\dot{\phi}^2) - \frac{\Lambda}{2} \tag{27}$$

$$\left(\frac{\dot{R}}{R}\right)^2 + 2\frac{\ddot{R}}{R} + \frac{k}{R^2} = -p - \frac{1}{2}\,\mathrm{e}^{\frac{\lambda}{2}\dot{\phi}^2} - \frac{\Lambda}{2} \tag{28}$$

$$\ddot{\phi}(1 + \lambda\dot{\phi}^2) + 3\frac{\dot{R}}{R}\dot{\phi} = 0. \tag{29}$$

Here, and in the sequel, we use the following convention: $\kappa = 8\pi G = 1, c = 1$. Let us consider some particular cases.

4.1. A flat FL-universe without matter

$$3\left(\frac{\dot{R}}{R}\right)^2 = \frac{1}{2}\,\mathrm{e}^{\frac{\lambda}{2}\dot{\phi}^2}(-1 + \lambda\dot{\phi}^2) - \frac{\Lambda}{2}$$

$$\left(\frac{\dot{R}}{R}\right)^2 + 2\frac{\ddot{R}}{R} = -\frac{1}{2}\,\mathrm{e}^{\frac{\lambda}{2}\dot{\phi}^2} - \frac{\Lambda}{2}$$

$$\ddot{\phi}(1 + \lambda\dot{\phi}^2) + 3\frac{\dot{R}}{R}\dot{\phi} = 0.$$

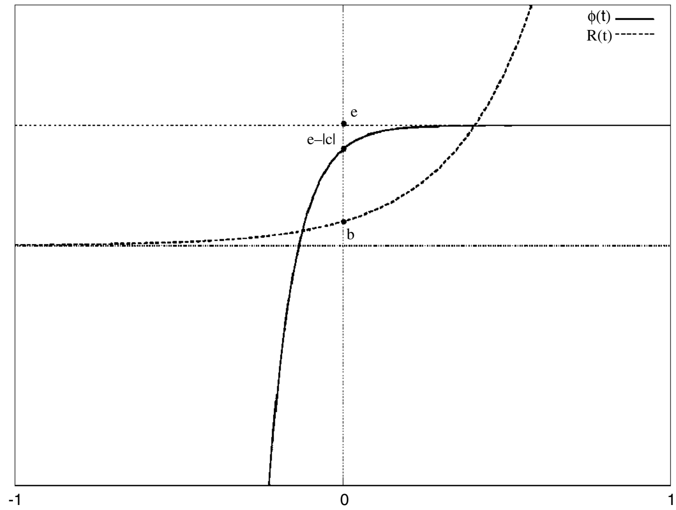

Figure 2. Behaviour of the expansion factor $R(t)$ and the exponentially harmonic map $\phi(t)$ on an Euclidean Friedmann–Lemaître universe in the Einstein frame ($\lambda \approx 0$; $\Lambda = 0$; $e > |c| + b$ and $a = -(\frac{3}{2})^{1/2}$).

Let us define $y = \dot{\phi}$ and let $H = \frac{\dot{R}}{R}$ be the Hubble constant. We check that $\dot{H} + H^2 = \frac{\ddot{R}}{R}$, then the equations above define a dynamical system:

$$\dot{H} = -\frac{\lambda}{4} y^2 \, e^{\frac{\lambda}{2} y^2} \tag{30}$$

$$\dot{y} = -\frac{3Hy}{1 + \lambda y^2} \tag{31}$$

subject to the constraint

$$H^2 = \frac{1}{6} \left((-1 + \lambda y^2) \, e^{\frac{\lambda}{2} y^2} - \Lambda \right). \tag{32}$$

Let us derive (32) and, using (31), let us compare the result with (30). We see that the preceding equations are compatible. We are thus led to solve the following equation:

$$\frac{\ddot{y}}{y}(1 + \lambda y^2) - \left(\frac{\dot{y}}{y} \right)^2 (1 - \lambda y^2) = \frac{3}{4} \lambda y^2 \, e^{\frac{\lambda}{2} y^2}. \tag{33}$$

For very small values of λ, this becomes

$$\frac{\ddot{y}}{y} - \left(\frac{\dot{y}}{y} \right)^2 \approx 0. \tag{34}$$

This gives the solutions

$$\phi(t) = c \, e^{at} + e \qquad R(t) = b \, e^{Ht}$$

where a, c are constants with the same sign, b is a positive constant and e is an arbitrary constant. $H = -\frac{a}{3}$ (H is a first integral when $\lambda = 0$). Equation (32) allows us to write $\Lambda = -1 - \frac{2a^2}{3}$ (see figure 2).

It is interesting to note here that the coupling of ϕ with the gravitational field can make ϕ regular at $t = 0$ in this Euclidean case, which was not the case in the uncoupled situation (see section 3).

Let us return to the case where λ is an arbitrary constant. We define $z = \dot{y}/y$, equation (33) leads to

$$\dot{y} = zy \qquad \dot{z} = \frac{2\lambda y^2}{1+\lambda y^2}\left(\frac{3}{8}\,\mathrm{e}^{\frac{\lambda}{2}y^2} - z^2\right)$$

which is a dynamical system with a fixed point $(y, z) = (0, 0)$. If we put $u = y^2$ we get

$$\dot{u} = 2zu \tag{35}$$

$$\dot{z} = \frac{2\lambda u}{1+\lambda u}\left(\frac{3}{8}\,\mathrm{e}^{\frac{\lambda}{2}u} - z^2\right). \tag{36}$$

The Hubble constant can be deduced from the numerical integration of this dynamical system. From (32) we get

$$H = \pm\left[-\frac{1}{6}\left((1-\lambda u)\,\mathrm{e}^{\frac{\lambda}{2}u} + \Lambda\right)\right]^{1/2}. \tag{37}$$

Around $u \approx 0$ (i.e. $\phi = \text{constant}$) we can neglect terms of order greater than or equal to 2. Using (35) and (36) we have at first order

$$u \approx -\frac{1}{2\lambda}\ln\left(|\lambda|\left(-\frac{3}{4} + 2z^2\right)\right) + u_o. \tag{38}$$

As $u = y^2$, we have to suppose that (for $\lambda > 0$): $-\frac{3}{4} + 2z^2 < \frac{1}{\lambda}$ when $u_o = 0$.

4.2. A curved F–L universe with matter: $p = \omega\rho$

Using (29), we can determine $R(t)$ easily

$$\frac{1}{R^3} = \alpha\dot{\phi}\exp\frac{\lambda}{2}\dot{\phi}^2. \tag{39}$$

Let us multiply (27) by ω and add (27) and (28). We get

$$\left(\frac{\dot{R}}{R}\right)^2(3\omega+1) + \frac{2\ddot{R}}{R} + \frac{k}{R^2}(3\omega+1) + \frac{1}{2}\,\mathrm{e}^{\frac{\lambda\dot{\phi}^2}{2}}(1+\omega-\lambda\omega\dot{\phi}^2) + \frac{1}{2}\Lambda(\omega+1) = 0. \tag{40}$$

Using (29), we find

$$\frac{\ddot{R}}{R} = -\frac{\dddot{\phi}}{3\dot{\phi}}(1+\lambda\dot{\phi}^2) + \frac{\ddot{\phi}^2}{9\dot{\phi}^2}(\lambda^2\dot{\phi}^4 - \lambda\dot{\phi}^2 + 4). \tag{41}$$

Then, by (27) and using the preceding notation, we can write the equation

$$\ddot{y} = \frac{\dot{y}^2}{y}\left(\frac{(3+\omega) + 2\lambda\omega y^2 + \lambda^2(1+\omega)y^4}{2(1+\lambda y^2)}\right) + \frac{3}{4}\frac{y\Lambda(\omega+1)}{1+\lambda y^2}$$
$$+\frac{3}{4}\,\mathrm{e}^{\frac{\lambda}{2}y^2}\frac{(1+w)y - \lambda\omega y^3}{1+\lambda y^2} + \frac{3}{2}\,\mathrm{e}^{\frac{\lambda}{3}y^2}\alpha^{2/3}y^{5/3}\frac{k(3\omega+1)}{1+\lambda y^2}. \tag{42}$$

Using the conservation law $\nabla_\mu T^{\mu\nu}_{\text{mat}} = 0$ and equation (29), we find

$$\frac{\dot{\rho}}{\rho(1+\omega)} = -3\frac{\dot{R}}{R} = \frac{(1+\lambda)\dot{y}}{y}. \tag{43}$$

An example of numerical solutions is given in figures 3–5.

In figure 3, we can see the typical behaviour of a Friedmann–Lemaître universe whose expansion is driven by exponentially harmonic maps and matter or radiation. This evolution

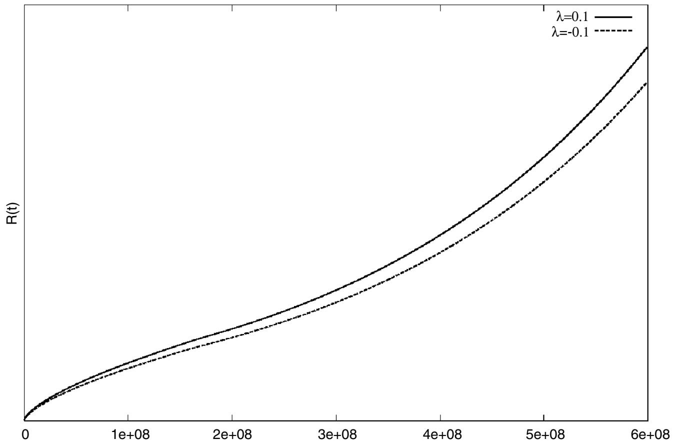

Figure 3. Expansion factor computed from equations (49) and (39) illustrating the behaviour of a hypothetical universe filled with matter ($\omega = 0$) and an exponentially harmonic map ($\dot{\phi}_0 = 1, \ddot{\phi}_0 = -5, \omega = 0, k = 0$).

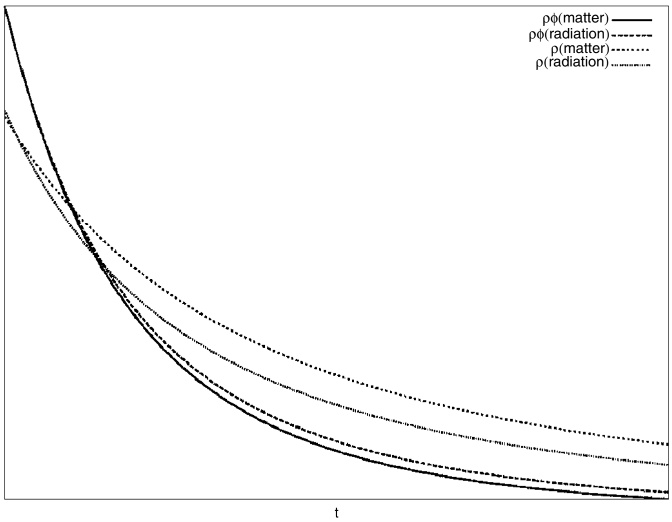

Figure 4. Evolution of energy densities of matter, radiation and the coupled scalar field. The arbitrary constants arising in the integration of equation (43) have been chosen in order to draw all the curves in the same plot ($\dot{\phi}_0 = 1, \ddot{\phi}_0 = -5, \lambda = 0.1, k = 0$).

knows three different ages: first, the universe behaves in the usual way under the presence of matter or radiation, then its characteristic deceleration stops and finally, when the exponentially harmonic map ϕ becomes too small, the expansion accelerates (see equation (39)). But this is also the typical behaviour of a universe that is dominated eventually by a positive cosmological constant or something that acts like it, such as a potential related to a scalar field (quintessence). Nevertheless, the careful reader has already noted that this particular behaviour is only due to the exponentially harmonic feature of the scalar field ϕ and its particular initial conditions $\dot{\phi}_0$ and $\ddot{\phi}_0$. This proves again that these interesting features, suggested among others by the

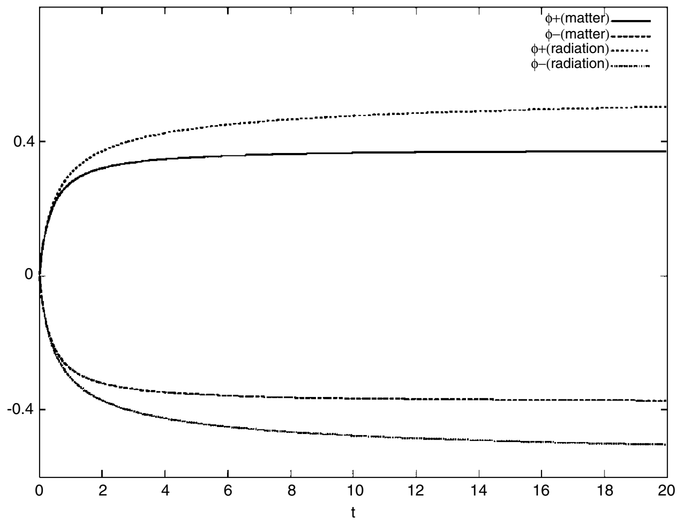

Figure 5. Exponentially harmonic maps $\phi = \pm\sqrt{u}$ solution of equation (49), for $\omega = 0$ (dust-dominated universe) and $\omega = 1/3$ (radiation-dominated universe) ($\dot{\phi}_0 = 1, \ddot{\phi}_0 = -5, \lambda = 0.1, k = 0$).

observations of type Ia supernovae, arise naturally in the context of exponentially harmonic maps without requiring an *ad hoc* potential for the scalar field.

In figure 4, we have represented the evolution of the energy densities of matter, radiation and the scalar field coupled to it. In the very early ages of the universe[5], the dominant contribution to the total energy is due to ϕ. But the universe expands just as if there was only the usual matter and radiation. As this expansion goes on, the energy density of the scalar field ρ_ϕ goes below the density of matter and radiation, making the expansion accelerating when the field ϕ become tiny (see (39)). This would correspond to the well-known '*cosmic coincidence*'. Note that we have chosen the arbitrary constants arising in the integration of equation (43) in order to get this peculiar model of expansion. The present study is purely qualitative, more work is under way to construct a quantitative model that could be compared, for example, with those of quintessence.

In figure 5, we have represented the evolution of the scalar field ϕ in order to satisfy equation (42) both in the presence of matter ($\omega = 0$) and radiation ($\omega = 1/3$).

5. Conclusions

The main interest of the use of exponentially harmonic maps is the fact that the Lagrangian (26) is a generalization of the bosonic string Lagrangian (with only a dilatonic field) written in the Einstein frame [9]. Indeed, if $\lambda \approx 0$, the Lagrangian (26) tends to this bosonic string Lagrangian, if $\Lambda = -1 - \Lambda_o$, where Λ_o is a small cosmological constant. But mathematically, this limit $\lambda \to 0$ is highly problematic due to the fact that, unlike the usual harmonic maps, exponentially harmonic maps have no good invariance property under homothetic changes of the field, as was shown in the introduction.

[5] It is interesting to note that here there is a huge difference in the time scales of figures 3 and 4, about seven orders of magnitude.

An interesting open question is whether solutions of equations deriving from (26) admit duality symmetries. We are in the process of studying this question.

After having completed the paper, we have observed that exponentially harmonic maps coupled with gravity are in fact a natural generalization of the scalar field used by Armendáriz-Pićon, Damour and Mukhanov in the context of the so-called '*k-essence*' and '*k-inflation*' (inflation driven by non-quadratic kinetic terms in the Lagrangian density). Apparently, the case we considered here was not studied in this framework.

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[9] See, for example, Lidsey J E, Wands D and Copeland E J 1999 Superstring cosmology *Preprint* hep-th 9909061